Probabilistic Methods in Combinatorics

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Assignment 9

To solve for the Example class on 29th April. Submit the solution of Problem 2 by Sunday 27th April if you wish feedback on it. Some hints will be given on Friday 18th April.

Problem 1. Let $x_1, \dots, x_m$ be boolean variables. In mathematical logic, a *literal* is an atomic formula x_i or its negation $\neg x_i$. A *formula* φ is a k-*CNF* (conjunctive normal form) if it is a conjunction of *clauses* $C_1, C_2, \dots, C_n$ (i.e. $\varphi = C_1 \wedge C_2 \wedge \dots \wedge C_n$) where each clause C_i is a disjunction of k literals $y_{i,1}, \dots, y_{i,k}$ (i.e. $C_i = y_{i,1} \vee y_{i,2} \vee \dots \vee y_{i,k}$) corresponding to different variables. The formula φ is said to be *satisfiable* if it can be made true by assigning appropriate logical values (i.e., true/false) to its variables. This means that, under such assignment, for every clause in φ at least one of its literals is true. For example, the 2-CNF

$$(x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_2) \wedge (x_1 \vee \neg x_2)$$

is satisfiable because for the assignment $(x_1, x_2) \mapsto (\text{true}, \text{false})$ we get

$$(\text{true} \vee \text{false}) \wedge (\text{false} \vee \text{true}) \wedge (\text{true} \vee \text{true}) = \text{true}$$

whereas the 2-CNF

$$(x_1 \vee x_2) \wedge (\neg x_1 \vee x_2) \wedge (x_1 \vee x_3) \wedge (\neg x_1 \vee x_3) \wedge (\neg x_2 \vee \neg x_3)$$

is easily seen not to be satisfiable. The problem of deciding whether a k-CNF is satisfiable is called the k-SAT problem. For $k \geq 3$ this problem is known to be NP-complete. Show that any k-CNF in which every variable appears in at most $2^{k-2}/k$ clauses is satisfiable.

Problem 2. Let H be a d-uniform d-regular hypergraph (i.e. each edge consists of d vertices and each vertex is in precisely d edges). Show that if $d \geq 6$ then the vertices of H can be coloured with two colours, red and blue, such that every edge is nearly balanced, meaning that the number of red and blue vertices in every edge differ by at most $\sqrt{6d \log d}$.

Problem 3. Let $G = (V, E)$ be a simple graph and suppose each $v \in V$ is associated with a set $S(v)$ of colours of size at least $10d$, where $d \geq 1$. Suppose, in addition, that for each $v \in V$ and $c \in S(v)$ there are at most d neighbours u of v such that c lies in $S(u)$. Prove that there is a proper colouring of G assigning to each vertex v a colour from its class $S(v)$.